electronics

MDPI

Article

Sub-Diffraction Photon Trapping: The Possible Optical Energy Eigenstates within a Tiny Circular Aperture with a Finite Depth

Ozan Turhan Gündüz [1], Erich Leitgeb [2,*] and Cumali Sabah [1]

[1] Department of Electrical and Electronics Engineering, Middle East Technical University—Northern Cyprus Campus (METU-NCC), Kalkanli, Guzelyurt 99738, TRNC/Mersin 10, Turkey; ozan.gunduz_01@metu.edu.tr (O.T.G.); sabah@metu.edu.tr (C.S.)
[2] Institute of Microwave and Photonic Engineering, Graz University of Technology, Inffeldgasse 12/I, 8010 Graz, Austria
* Correspondence: erich.leitgeb@tugraz.at

Abstract: One of the challenging riddles that is set by light is: do photons have wavefunctions like other elementary particles do? Wave–particle duality has been a prevailing fact since the beginning of quantum theory thought; in electromagnetism, light is already a kind of undulation, so what about the waves of probability then? Well, Quantum Field Theory (QFT) has a rigorous explanation and supports the idea when they are considered as fields of particles via second quantization; they do have wavefunctions of probability, and it does not have anything to do with the regular oscillations. They can be related to the energy and momentum signatures of harmonic oscillations, resembling an imitation of the behavior of a classical harmonic oscillator, which then has a wavefunction to solve the corresponding time-independent Schrödinger equation. For the last half century, electrical engineering has owned the best out of these implications of Quantum Electrodynamics (QED) and QFT by engineering better semiconductor techniques with finely miniaturized transistors and composite devices for digital electronics and optoelectronics fields. More importantly, these engineering applications have also greatly evolved into combined fields like quantum computing that have introduced a completely new and extraordinary world to electronics applications. The study takes advantage of the power of QFT to mathematically reveal the bosonic modes (Laguerre–Gaussian) that appear in a sub-diffraction cylindrical aperture. In this way, this may lead to the construction of the techniques and characteristics of room-temperature photonic quantum gates which can isolate photon modes under a diffraction limit.

Keywords: quantum field theory; photon trapping; canonical quantization; quantum computing; quantum electronics applications; photon trapping; orbital angular momentum

check for updates

Citation: Gündüz, O.T.; Leitgeb, E.; Sabah, C. Sub-Diffraction Photon Trapping: The Possible Optical Energy Eigenstates within a Tiny Circular Aperture with a Finite Depth. *Electronics* **2023**, *12*, 4832. https://doi.org/10.3390/electronics12234832

Academic Editor: Nakkeeran Kaliyaperumal

Received: 29 September 2023
Revised: 16 November 2023
Accepted: 22 November 2023
Published: 30 November 2023

Copyright: © 2023 by the authors. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (https://creativecommons.org/licenses/by/4.0/).

1. Introduction

Optical confinement (trapping) of photons can provide great potential for testing their quantum structures under a standard diffraction limit where Fraunhofer and Fresnel diffractions are not adequate explanations to reveal what is really going on. Could they exist in a place where classical electromagnetism has no useful clue as to their presence? This question may be answered by the photon wavefunction concept through QFT and has been addressed by few works in the literature [1,2]. Not only for the sub-diffraction region, but also for the situations where the wavelength of light is smaller than the aperture diameter, it is possible to interpret the reasons and dynamics of Fraunhofer diffraction in detail under the assumptions of the Copenhagen interpretation [3]. Once a single photon is taken into consideration, its momentum is definite, whereas its position is not due to its wave nature. Even if it is a particle, it should be handled as a wave–corpuscle which spreads all over the space or at least in a certain direction. Not to mention, the spin-one nature of a photon with a spin–momentum coupling also makes position and spin measurement unclear, and the localization of a single photon becomes impossible. This fact about photon

localization, unlike that of fermions or any other sub-atomic particles, has driven the photon wavefunction subject into a long-lasting debate since the first half of the twentieth century. It has been illustrated before that the expectation value for the position of a photon with non-zero spin cannot be similar to that of fermions, or even possible [4]. Further research also has identified the issue of photon position representation [5,6]. Despite light's unusual properties, unlike those of any other phenomena, recent studies [7,8] have identified the existence of a localized set of independent bases which can give proper position measurements, and, in fact, it has been indicated also by Hawton and Debierre in 2019 [9] that the position operator of a photon has rather a curvilinear cylindrical symmetry that actually empowers our localization approach within a cylindrical aperture. Based upon the Newton–Wigner rigorous localization of non-zero spin particles [10], there have recently been various efforts [11–14] to enlarge the position operator models in order to also fit in the photon picture by different combinations of inner products to generate appropriate wavefunctions.

The photon wavefunction concept is in fact a prevailing subject which is usually handled by the help of second quantization [15]. This is a bridge between Maxwell's equations and the non-relativistic form of the Dirac equation upon stationary-state assumptions. Even in non-relativistic cases, the formation of the wavefunction must obey certain relativistic rules. It is not allowed for the photon wavefunction to spatially expand outside the light cone. Unlike other sub-atomic particles, photons are bound to spread non-stop at a constant rate. As a result, the wavefunction must be spatially distributed by means of space–time in such a causal way that relativistic contradictions do not occur.

For the situations where the gravitational field and/or any kind of acceleration/deceleration forces the space–time metric into a non-unity formation, the system Hamiltonian must still support the gauge invariance of the quantized field parameters even under the geodesics dictated by this metric. The d'Alembertian of the resulting energy eigenstates must vanish, $\eta^{\mu\nu}\partial_\nu\partial_\mu\psi(q,t) = 0$, and satisfy the Klein–Gordon equation for massless cases with its fully established space–time pattern. As the problem is simply to find the viable trajectory of light in the presence of space–time disturbance, the time signature of the wavefunction comes into play as another ruling factor. For accelerated/decelerated frames of references, one may consider the quantization of the field in the Rindler space since the space–time worldline turns out to be hyperbolic shaped. A successful study of non-inertial quantization, also in good agreement with the implications of the Fulling–Davies–Unruh effect, exists in the literature [16]. In this respect, the time dependence of the wavefunction must be handled carefully, and the system Hamiltonian is supposed to be evaluated accordingly. Since the dynamics of non-inertial reference frames are far more complicated compared to in an inertial case, this is a subject that is out of scope for our study.

The fact that light is always in motion complicates the mathematical and conceptual modeling of the particle dynamics of photons, and, therefore, the locality of classical electromagnetism has always been a challenge for the clipping of a wavefunction, obeying non-local quantum dynamics, onto a photon. Additionally, the idea that even an individual light quantum follows through the uncertainty principle when examined a priori casts doubt on the existence of a photon wavefunction.

Nevertheless, a single photon is usually imagined as an electromagnetic undulation with an unimaginably dimmed intensity (hypothetically) at an exact energy level with a well-defined momentum. This view does not contradict Maxwell's picture, and it is possible for a single photon to generate an interference pattern when it interacts with an observer [17,18]. The photon lands on a different point over the whole interference pattern in each trial, which basically suggests that once the photon interacts as a particle, it obeys a probability wavefunction even if the interference pattern can be linked to its electromagnetic nature. This is a solid illustration of the wave nature of light quanta literally extending beyond the electromagnetism. The most viable guess would be that the localizable and invariant wavefunction does exist, provided that it is cylindrically

confined and light quanta interact with an observing screen at the end of the cavity. Thus, the foundations of photoelectric effect in conjunction with this situation further push us towards the construction of the wavefunction idea with regard to second quantization. Nevertheless, the significant and elegant symmetry exhibited by Maxwell's equations in the resemblance of the quantized field equations slants the theory towards the quantum versions of the field expressions.

Consequently, the study mainly contributes to the following issues: (a) the possible energy eigenstates of confined photons in the cases where they have local wavefunctions only in a sub-diffraction cylindrical aperture (an open-ended cavity; from now on, we will call it just "cavity"), (b) proper normalization of the wavefunctions under confinement to reveal photon probability densities within the cavity, and (c) the possible applications which the consequences of the study may address and which may be useful for quantum electronics.

The article sections are structured as follows: the comparative analysis between the semi-classical approach to investigate the passage of light through a sub-diffraction cylindrical aperture, in Section 2, and the canonical quantization method, in Section 3, to indicate how useful the second quantization is over rudimentary techniques. The distribution of bosonic probabilities in the cavity is also demonstrated along with the discussion of probable applications in Section 4.

2. Semi-Classical Field Quantities within the Cavity

The transverse modes in a waveguide cavity, regardless of the shape, are well investigated by classical electromagnetism. Multiple modes, including some hybrid modes, propagate simultaneously, and their frequency signature is continuous even if the modes are themselves discrete. The electric intensity is typically in the form of Bessel functions for the circular waveguides, and, of course, the propagation has some certain bandwidths as the higher modes start to kick in. No matter what, the frequency windows are composed of a continuum where it is possible for the base mode to propagate any frequency. Once we consider this waveguide as a nano-hole of a potential well (we use this terminology to indicate that it is just a cavity to trap photons, not exactly a potential well which can only confine fermions with non-zero charge—this is the easiest way to conceptually define a circular aperture with a thickness), things become weird as this continuum no longer exists due to the uncertainty relation. Only discrete energy levels are possible, which is never addressed by classical electromagnetism.

But before getting into this consideration, let us investigate the situation rather intuitively. Let us assume a visible light is incident on a surface of a potential volume which has a nano-hole that photons can exist within. For this, we have to point out the following physical assumptions for the apparatus:

- The whole system (including the light source) is an inertial reference frame;
- The light source is a quasi-monochromatic laser light with a Gaussian single-mode beam distribution (a stable laser source with almost coherent states);
- The intensity of the laser source is well adjusted to suppress any non-linear effect along with an acceptable level of quantum noise;
- The diameter of the hole is sub-diffraction but larger than the wavelength such that the photon distribution over the entrance is uniform.

The procedure we will follow here assumes a Gaussian beam of $E_0 e^{-\frac{\rho^2}{w_0^2}} e^{-j(kz-\omega t+\theta)} \hat{\boldsymbol{\epsilon}}_1$, which results from the infinite beam radius assumption where the Rayleigh range becomes very large compared to the traversed distance of the light, and beam spot size is closer to initial spot size $w_0/w(z) \approx 1$ for this approximation. There has intuitively to be no variations over the tiny circular hole, i.e., the entire cross section of the circular aperture of the potential well has the same field intensity as we would anticipate since the beam size of the source is huge in comparison. Additionally, the wave propagation based on dispersion

perhaps is negligible over a single aperture due to its size as we consider that even the most monochromatic light cannot be a single mode of frequency in practice.

The total Hamiltonian of the system is not an easy parameter to find out as the shear forces within the cavity make it impossible to know exactly what the system momentum is, which may be highlighted by Maxwell's stress tensor at a certain accuracy. The assumption that incoming light carries all the energy and momentum would hold, a fortiori, to show at least where the energy comes from. But in terms of the quantum mechanical view, this prediction is not very helpful.

Nonetheless, due to the sub-diffraction quantum well, the intensity of the light will be dimmed at a degree of about $1/4(D/\lambda)^2$, where λ is the free-space wavelength, and D is the diameter of the hole [1]. So, the analysis will be based on a maximum transmission efficiency of about 36% of the entire field intensity. Indeed, this number represents the total statistical distribution of the photon field not that of the individual quanta.

Field Quantities in Circular Quantum Well via Semi-Classical Approach

Classically, the potential wells are represented by material boundaries where waves must meet certain conditions, which allows discrete modes or cut-off frequencies to be presented. Intuitively, as the propagation takes place in the longitudinal direction, it will not change in the curvilinear aperture, and the standing modes must be observed in the transverse plane. How it evolves in the potential well is yet a mystery, but it should fill the following wave equation for the electric field as follows:

$$\partial_\rho{}^2 E + \frac{1}{\rho}\partial_\rho E + \frac{1}{\rho^2}\partial_\phi{}^2 E + \partial_z{}^2 E + k^2 E = 0 \tag{1}$$

This is the open form of the curvilinear Laplacian operator acting on the electric field vector. Assume a separable solution, $E_1(\rho)E_2(\phi)E_3(z)$, for (1) due to the nature of electromagnetic undulations and the geometry of the well. The most difficult component of (1) is the transverse part, which happens to be as follows:

$$\rho^2 E_1{}'' + \rho E_1{}' + \rho^2 k^2 E_1 - \zeta^2 E_1 = 0 \tag{2}$$

Here, ζ is the second-order slope of $E_2(\phi)$, which is the angular momentum constant, and helps to simplify the equation. The parameter ρ creates the difficulty, but its variation $\Delta\rho \approx 0$ over the hole is extremely tiny. Therefore, the parameter can be assumed to be almost constant in the nanometric window of Equation (2). Also, in reality, the value of ρ will be small compared to the propagation constant and angular momentum. Once the whole equation in (2) is divided by ρ, the coefficients of lower-order terms outweigh the highest-order derivative; as a result, the equation becomes first order, which is rather more trivial to solve. Therefore, the solution $E_1(\rho) = C_1 J_\zeta(k\rho) + C_2 Y_\zeta(k\rho)$ involves the first and second kinds of Bessel functions, whose indexes depend on the angular momentum. The results are almost in line with circular waveguides; however, there is a certain problem at the center where the Bessel function of the second kind becomes negatively infinite. There must be at least a singularity killer weighing function in the solution, which sadly does not exist in this form of (2), with nanometric approximation.

The angular and longitudinal solutions are typical harmonic and propagation expressions, respectively, and have no contribution to null at the center whatsoever. They are also in line with the natural diffraction results, with slight differences, but cannot explain well what happens at the center. Briefly, the result is not normalizable and does not support a rigorous wave formation in the well. Knowing that bosons do support quantum harmonic oscillator energy signatures, which resemble their particle properties well, the quantization of the wave based upon this idea is supposed to give some concrete results. To do so, we must accept the assumption that photons must also obey probability wavefunctions—apart from their electromagnetic wave nature—as any other sub-atomic particle does. We are not fully comfortable in this assumption; however, the Hamiltonian of the quantum harmonic

oscillator rigorously results in the bosonic energy quantum, which is simply a very useful approach by means of quantum electronics applications.

3. The Possible Energy Eigenstates by Second Quantization

Without going into details about the well-known harmonic oscillator Hamiltonian and how it is related to first quantization with canonical commutation relations, our focus will be the curvilinear Schrödinger equation since the system is at rest, and no relative motion of fermions exists in this case. This will extract the possible probability distribution of photons within the cylindrical aperture (considering it only as a potential well) easily with the non-relativistic Schrödinger equation when there is not any material around.

The Hamiltonian operator in a generalized curvilinear coordinate system can be represented by the following differential form with energy quantum:

$$\hat{\mathcal{H}} = \frac{\hbar\omega}{2}\left[-\partial_q{}^2 - \frac{1}{q}\partial_q - \frac{1}{q^2}\partial_\phi{}^2 - \partial_{\mathbb{z}}{}^2 + q^2 + \mathbb{z}^2\right] \tag{3}$$

Here, $q = \sqrt{\omega/\hbar}\rho$ and $\mathbb{z} = \sqrt{\omega/\hbar}z$ represent the non-dimensionalized space coordinates of the radial and longitudinal direction, respectively. At the same time, they are the generalized coordinates defined to extract the pure energy and momentum characteristics of the system. It is hard to separate the angular momentum component from the radial part in (3), but it can be considered as a constant like the previous case. Assuming a separable solution, $\Psi(q,\phi,\mathbb{z}) = \psi_R(q)\psi_A(\phi)\psi_L(\mathbb{z})$ will be helpful as the eigenvalue equation $\partial_\phi{}^2\psi_A(\phi) = -m_l{}^2\psi_A(\phi)$ for the angular part becomes consistent. Since the partial derivative must be independent from the radial direction, the operator equation $\partial_\phi{}^2 = -m_l{}^2$ must follow as we know the angular momentum quantum number is the eigenvalue of the longitudinal angular momentum. Locating the angular quantum number in (3) and using the result from the complete eigenvalue equation gives the following differential equations:

$$\partial_q{}^2\psi_R(q) + \frac{1}{q}\partial_q\psi_R(q) - \left(\frac{m_l{}^2}{q^2} + q^2 + \kappa_R\right)\psi_R(q) = 0 \tag{4}$$

$$\partial_\phi{}^2\psi_A(\phi) + m_l{}^2\psi_A(\phi) = 0 \tag{5}$$

$$\partial_{\mathbb{z}}{}^2\psi_L(\mathbb{z}) - \left(\mathbb{z}^2 + \kappa_L\right)\psi_L(\mathbb{z}) = 0 \tag{6}$$

The coefficients in (4) and (6), $\kappa_R = -\frac{2E_R}{\hbar\omega}$ and $\kappa_L = -\frac{2E_L}{\hbar\omega}$, represent the energy eigenvalues of the radial and longitudinal directions, respectively. On the one hand, Equations (5) and (6) are familiar to solve; on the other hand, the curvilinear equation in (4) is a degenerate hypergeometric differential equation. Luckily, the intermediate weighing solution in the form of Poisson distribution $\psi_{int}(q) = q^{m_l}e^{-q^2/2}\psi_R(q)$ fits nicely into (4) because the singularity at $q = 0$ vanishes for the positive values of the radius for sure. Along with the intermediate solution, Equation (4) becomes a confluent hypergeometric equation, of which one of the possible solutions is a famous generalized Laguerre polynomial along with Gamma function factors as follows:

$$q\frac{d^2\psi_R(q)}{dq^2} + \left(2m_l + 1 - 2q^2\right)\frac{d\psi_R(q)}{dq} - (2m_l + 2 + \kappa_R)q\psi_R(q) = 0 \tag{7}$$

With the solution

$$\psi_R(q) = C\frac{\Gamma(1 - 1/4[2m_l + 2 + \kappa_R])\Gamma(m_l + 1)}{\Gamma(-\kappa_R + m_l/2 + 1/2)}L^{m_l}_{-1/4[2m_l+2+\kappa_R]}\left(q^2\right) \tag{8}$$

The solution can be alternatively represented by Kummer's Functions of the First Kind since the generalized Laguerre polynomials along with Gamma function coefficients in the form revealed in (8) are basically one of the possible expansions of generalized hypergeometric series. Alternative approaches and representations of the solutions of the differential equation in (7) have been also investigated in the literature [19]. It is highly expected for the index of Laguerre polynomials to be integers, which leaves us no choice but to consider the equality $n = -1/4[2m_l + 2 + \kappa_R]$ since it resolves into the total energy quantum of curvilinear system $E_R = \hbar\omega(2n + m_l + 1)$ elegantly. Therefore, we can call n as the principal quantum number n_q of the radial direction and rewrite the solution in (8) as follows:

$$\psi_R(q) = C\frac{n_q!\Gamma(m_l+1)}{\Gamma(n_q+m_l+1)}L_{n_q}^{m_l}\left(q^2\right) = C_{n_q,m_l}L_{n_q}^{m_l}\left(q^2\right) \tag{9}$$

The actual difficulty is to normalize the wavefunction, which is to find the normalization constant C_{n_q,m_l} of energy eigenstates. Since the Laguerre polynomials are undoubtedly orthonormal and can form a proper basis in Hilbert space, the wavefunctions must also be square integrable. Then, the total probability integral within the aperture will be as follows, by applying $q^2 = Q$ coordinate transformation:

$$\frac{{C_{n_q,m_l}}^2}{2}\int_0^\infty Q^{m_l}e^{-Q}\left[L_{n_q}^{m_l}(Q)\right]^2 dQ = 1 \tag{10}$$

Removing Laguerre squared by separating it as ${C_{n_q,m_l}}^2/2n_q!\int_0^\infty \frac{d^{n_q}}{dQ^{n_q}}\left(Q^{n_q+m_l}e^{-Q}\right)L_{n_q}^{m_l}(Q)dQ$ will ease the integration process of integration by the following parts:

$$= \frac{{C_{n_q,m_l}}^2}{2n_q!}\left[\frac{1}{n_q!}\left\{L_{n_q}^{m_l}(Q)\frac{d^{n_q-1}}{dQ^{n_q-1}}\left(Q^{n_q+m_l}e^{-Q}\right)\right\}\Big|_0^\infty - \frac{1}{n_q!}\int_0^\infty \frac{d}{dQ}L_{n_q}^{m_l}(Q)\frac{d^{n_q-1}}{dQ^{n_q-1}}\left(Q^{n_q+m_l}e^{-Q}\right)dQ\right] \tag{11}$$

The left-hand side just vanishes at both limits of integration, and the application of the procedure $n_q - 1$ times cancels the derivative and results in:

$$= \frac{{C_{n_q,m_l}}^2}{2n_q!n_q!}(-1)^{n_q}\int_0^\infty \frac{d^{n_q}}{dQ^{n_q}}\left[L_{n_q}^{m_l}(Q)\right]\left(Q^{n_q+m_l}e^{-Q}\right)dQ \tag{12}$$

The generalized Laguerre polynomials can be represented by infinite series as they include higher-order derivatives $L_{n_q}^{m_l}(Q) = \sum_{u=0}^{n_q}(-1)^u\binom{n_q+m_l}{n_q-u}\frac{Q^u}{u!}$, and this indicates that the series of Binomial coefficients cannot be sustained after taking the ${n_q}^{\text{th}}$ derivative up to the $u_{n_q}Q^{n_q}$ component. Thus, the integral after all would be:

$$\frac{{C_{n_q,m_l}}^2}{2n_q!}\int_0^\infty Q^{n_q+m_l}e^{-Q}dQ = 1 \tag{13}$$

This is a well-known integral resulting in $\Gamma\left(n_q+m_l+1\right)$, Euler's Gamma function, which, along with the normalization coefficient and factorial in (13) and in conjunction with the azimuthal normalization constant from the solution $e^{jm_l\phi}$ of (5), reveals the normalization constant beautifully as (we are combining both in C_{n_q,m_l} as it is not only for the radial direction):

$$C_{n_q,m_l} = \sqrt{\frac{2n_q!}{(n_q+m_l)!}} \tag{14}$$

Since the longitudinal component is in rectilinear coordinates, the energy eigenstates are found to be Hermite polynomials, and the normalization constant is well known as follows:

$$\psi_L(\mathbb{z}) = \frac{1}{\pi^{\frac{1}{4}}\sqrt{2^{n_z}n_z!}} e^{-\frac{\mathbb{z}^2}{2}} H_{n_{\mathbb{z}}}(\mathbb{z}) \tag{15}$$

Here, n_z is the principal quantum number and represents the number of energy quanta even if the aperture is open ended. The total probability distribution is then the square of the magnitude of the wavefunction, which is the multiplication of each component, as we previously mentioned.

$$|\Psi(q,\phi,\mathbb{z})|^2 = \pi^{-\frac{1}{2}} \frac{2n_q!}{(n_q+m_l)!2^{n_z}n_z!} q^{2m_l} e^{-q^2} L_{n_q}^{m_l}\left(q^2\right)^2 e^{-\mathbb{z}^2} H_{n_{\mathbb{z}}}(\mathbb{z})^2 \tag{16}$$

Notice that the angular component has no spatial contribution to total probability as the system geometry is assumed to have a perfect circular geometry. The only effect it has is on the amplitude of the probability, and one must not be confused here as to whether the open-ended aperture leads to a continuum in the longitude. The consideration here is just to reveal the true particle dynamics of photons within the cavity, and the confinement is just a conceptual one rather than being solid. The most important question here is: what will be the particle nature of light when it is in such a cavity? At least if photons obey wavefunctions, the probability distribution in (16) must be the probability distribution of the possible non-degenerate energy eigenstates in a cylindrical aperture since the results consistently form a proper orthonormal basis in Hilbert space. Multiplication of polynomials results in another polynomial which must also be linearly independent; however, they must only meet this criterion spatially, i.e., only in their corresponding space coordinate.

3.1. Visualization of Probability Distributions of Bosonic Modes within the Cavity

3.1.1. Weak Orbital Angular Momentum (OAM) Modes

In the previous section, we revealed the energy eigenstates of the bosonic modes, which are circularly symmetric in the azimuthal direction within the cylindrical aperture. Now, the probable propagation modes can be represented by the corresponding quantum numbers n_q, m_l and l, similar to the atomic orbitals. Inevitably, the longitudinal quantum number n_z governs the energy quanta in Hermite polynomials longitudinally.

We can, a priori, imply that the rings in the probability distributions illustrated in Figure 1a–d represent the weak OAM modes. It is important to note that, even if the ground energy appears with a higher probability, we named these modes as "Weak" to categorize the case $m_l = 0$ due to the faint appearance of the helices, as indicated in Figure 1, while the energy quantum increases. The entire probability distribution does not stand for the complete beam of light since a beam will have all of these energy eigenstates together in a statistical distribution. But a photon can be anywhere in these distributions with any polarization state and OAM state. When cylindrical geometries are involved in the passage, OAM modes are highly likely to occur. On the contrary, these probabilities are not in the connected helical shapes extending in the longitudinal direction like typical OAM modes.

The probability distributions for bosonic modes in the cylindrical well perfectly resemble the orbital-like photon energy levels. The ground-level amplitude exhibits a perfect Poisson distribution in Figure 1a that can only be observed in almost coherent radiation. While propagating, the photons may occupy this orbital in a manner almost identical to a 1 s orbital. These energy levels are the consequences of the confinement in the transverse direction by the cylindrical potential walls, which is why, as the transverse principal quantum number increases, some relatively faint annular structures begin to exist as a representation of OAM.

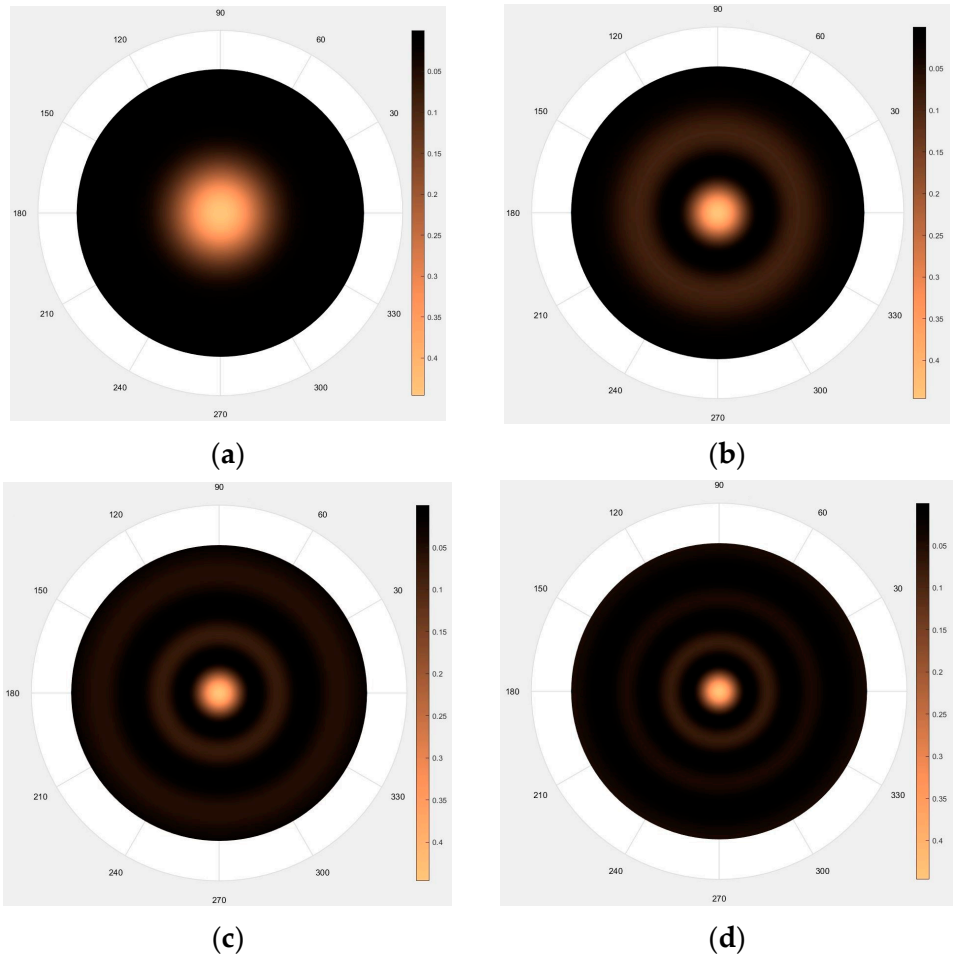
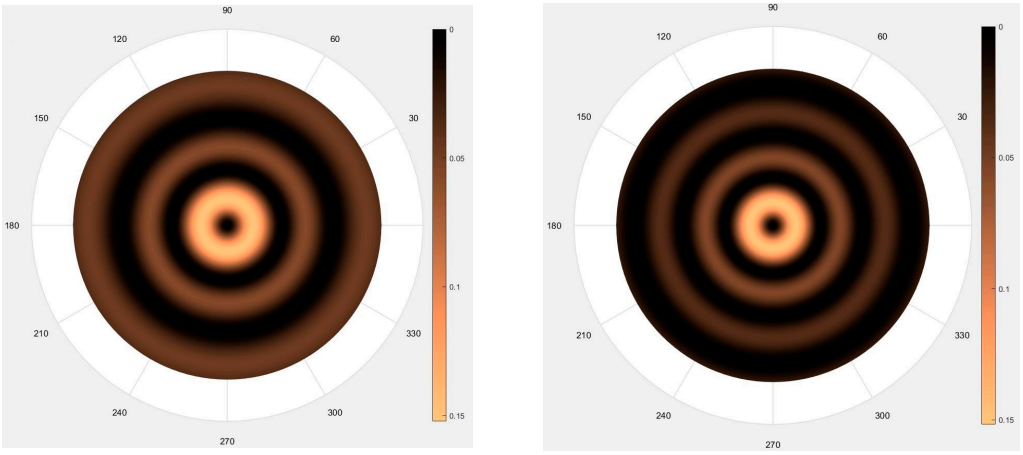


Figure 1. The probability distribution within the circular aperture. The results are over non-dimensionalized transverse coordinates ρ/ρ_0 with a maximum probability amplitude around 0.414 in: (**a**) $n_q = 0$, $m_l = 0$, $l = 0$ the aperture ground orbital (1st allowed energy level); (**b**) $n_q = 1$, $m_l = 0$, $l = 0$ aperture hybrid-1 orbital (2nd allowed energy level); (**c**) $n_q = 2$, $m_l = 0$, $l = 0$ aperture hybrid-2 orbital (3rd allowed energy level); (**d**) $n_q = 3$, $m_l = 0$, $l = 0$ (4th allowed energy level). As the transverse principal quantum number increases, the number of rings keeps increasing.

3.1.2. Strong OAM Modes

As is expected, no accumulation of bosons at the center of the aperture for the higher values of magnetic quantum number m_l takes place. The voids between the helices are simply the forbidden spaces for any photon, but, peculiarly, a photon can hop to any of the helices revealed in Figure 2a,b in any time via quantum tunneling.

Each figure illustrates multiple rings which are entangled with each other in each situation. Here, we see the well-defined OAM modes as a result of the Laguerre–Gaussian distributions. The probability intensity reduces as the principal quantum number increases since it means the distribution represents the photons with higher energy levels, which is less likely to occur. The main propagation through the aperture seems to be dominated by lower-energy photons as would normally be expected. If, somehow, there is a photon with a higher energy quantum than $n_q = 2$, for example, it will be most likely found at the very center ring, as Figure 2b suggests. The rising magnetic quantum number affects just the number of OAM rings. As the energy becomes higher, all the modes must collapse into a very small region at the middle like a dot and propagate as a stream.



(**a**) (**b**)

Figure 2. The probability distribution within the circular aperture. The results are over non-dimensionalized transverse coordinates ρ/ρ_0 with a highest probability amplitude occurring around 0.144 in: (**a**) $n_q = 2$, $m_l = \pm 1$, $l = 0, 1$, the 1st helical mode; (**b**) $n_q = 3$, $m_l = \pm 1$, $l = 0, 1, 2$, 2nd helical mode.

4. Discussion and Possible Application Areas

The results in the previous section as a result of (16) do not tell us anything about the polarization states of photons as they are determined by the field quantities. Thus, the field vector operators through second quantization must contain this information somehow. The polarization states must have the same amplitude as (16) but must be represented by a different ket notation. Also, the perpendicular linear polarization or right/left circular polarization states are from the Abelian group, and their number operators in clockwise and counterclockwise configurations do commute. In the context of polarization, the most useful parameters are Stokes parameters, through which quantum optical measurements make great sense. As usual, and in our case, the transverse components are the matter so the photon polarization is the ultimate quantity bridging the gap between optical and electronic dynamics.

The polarization states of photons may not only be manipulated via particle interactions through media but also can be whipped into shape via the presence of an external electric field stimulus. The electric field vector operator, governing bosonic fields, would definitely be affected by these fields, which themselves are nothing but vector operators. However, the singular definition attained for a photon cannot include the field orientation definition of the linear polarization; this variation would be reflected as the change in the polarization states. As a result, it would be possible to manipulate light passage through a material. This beautifully makes great opportunities to form opto-electronic quantum applications. The electrostatic potential must have an inevitable effect on the energy eigenstates of fermions, and this has been utilized to shift Fermi levels of materials to change light–matter interaction at quantum-level interband transitions for quantum electronics applications [20–24] for a while now.

One of the most attractive and popular applications of these effects has taken place with graphene interfaces lately [25–31]. The almost linear dispersion relation of graphene near Dirac points makes it easy to control the optical transmissivity of the layer as it turns out to be a unique semi-metal. The electron–hole accumulation is already located very near the Fermi level without any bandgap in between [32,33]. In the Hamiltonian matrix formulation of graphene, the shift in Fermi level is explicitly affected by the change in the constant electric field bias. This makes graphene one of the most suitable 2D materials for this kind of electro-optic control. Once the cylindrical potential well is formed, the OAM and Spin Angular Momentum (SAM) states of photons may successfully be adjusted. This

might lead to better-performing room-temperature quantum systems both for quantum information and measurements.

To give a realistic scenario, for instance, a thin film of a highly absorbing material at visible wavelengths can be decorated via an array of these sub-diffraction openings coated by a single or multi-layer bipartite graphene membrane. The tight-binding interaction Hamiltonian, and thus the total light energy that is allowed to pass the graphene membrane of each aperture, can be manipulated through an electrostatic potential distribution over the entire array. This would help in generating entangled photons as a consequence of boson–fermion interactions at the interface. Therefore, the creation of a qubit array, which we can call a photo-qubit film, is based on the manipulation of the polarization and spin states of the individual photon pairs and/or clusters. Building this film as an intermediate photon filter may give rise to certain technical benefits. In this way, significant room-temperature qubit systems could possibly be adopted, and the quantum information may be carried more effectively even at room temperature. Indeed, a meticulous and extensive further investigation of the subject related to the photo-qubit film idea is a must before moving on to application phases.

5. Conclusions

As we are so close to the ending of our study, it would be useful to gather some quick bullet points regarding the most important implications of the paper:

- The semi-classical techniques do not tell us exactly what is going on inside the cylindrical potential well when it is sub-diffraction sized;
- There are literally possible energy eigenstates inside the cylindrical potential well manifesting the visible photon sectors that are traversing through it;
- The quantum harmonic oscillator through second quantization results in consistent and square integrable bosonic wavefunctions within the potential well;
- The possible bosonic states are of generalized Laguerre–Gaussian type in the transverse direction (ρ and ϕ) and are of Hermite–Gaussian type in the longitudinal direction (z);
- These prove that photons can exist at discrete energy levels with their unique circular orbitals in even sub-diffraction circular apertures;
- Dominant probability distributions at lower-order unique OAM modes are observed in the mathematical manifestations of energy eigenstates;
- This configuration can be utilized via a carefully designed nanometric aperture as a room-temperature qubit employing photonics;
- It can also be utilized as a nanometric sensor or biosensor if the target to be sensed is ultra thin or is usually gaseous.

More importantly, we have achieved the ultimate purpose of the study: determination of whether bosonic states can appear, and, if so, what are the possible eigenstates within a cylindrical potential well which, in practice, is modeling the sub-diffraction aperture with a finite depth. This structure can be realized via a meticulous arrangement of molecules of some good opaque materials in a circular fashion such that the combined potentials of the lattices form a cylindrical confinement. Since the beginning of the study, the term potential well has been adopted; however, it is not possible to confine chargeless particles, unlike fermions. Thus, this can highly likely be achieved by using metal atoms with air inside the aperture in practice. But, in that case, surface plasmon polaritons (SPPs) will be formed on the internal surface which will alter the energy eigenstates of the bosonic fields. Despite its drawbacks, this topology will conceptually represent something close to a potential well to photons as there will be no photons able to exist within the metal parts. This is as if photons are forced to be trapped within the cylindrical space inside the sub-diffraction metal hole. Unluckily, one must deal with the reflections and scattering caused by electron–photon interactions.

Long story short, QFT suggests that it is possible to have light propagation in sub-diffraction apertures where standard diffraction models begin to be torn apart. The quantum harmonic oscillator model resolves elegantly in cylindrical systems and gives Laguerre–

Gaussian modes with well-defined OAM states. The question as to whether bosons can have wavefunctions like any other sub-atomic particle does seem to have been answered, at least to some degree; if they do, this is rigorously supported by QFT even in this kind of situation. As long as a very close design is realized within a semi-conductor device, these entangled modes of propagation that can be very useful in quantum information systems would definitely be yielded.

Author Contributions: Conceptualization, O.T.G.; methodology, O.T.G.; software, O.T.G.; validation, O.T.G. and C.S.; formal analysis, O.T.G.; investigation, O.T.G.; resources, O.T.G. and C.S.; data curation, O.T.G. and C.S.; writing—original draft preparation, O.T.G. and C.S.; writing—review and editing, O.T.G., C.S. and E.L.; visualization, O.T.G.; supervision, C.S. and E.L.; project administration, C.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by TU Graz Open Access Publishing Fund.

Data Availability Statement: All data underlying the results are available as part of the article and no additional source data are required.

Acknowledgments: This work was mainly carried out in the frame of a joined activity of the authors with different affiliations and resources. For this reason, the authors are very thankful to their home institutions the Middle East Technical University-Northern Cyprus Campus (METU-NCC) and Graz University of Technology (TUG) for the support of their research collaboration, the fruitful and efficient personal exchanges and the possibility to publish the results.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Bethe, H.A. Theory of Diffraction by Small Holes. *Phys. Rev.* **1944**, *66*, 163–182. [CrossRef]
2. Kakazu, K.; Kim, Y.S. Quantization of electromagnetic fields in a circular cylindrical cavity. *Phys. Rev. A* **1994**, *50*, 1830–1839. [CrossRef] [PubMed]
3. Fabbro, B.A. On the quantum theory of diffraction by an aperture and the Fraunhofer diffraction at large angles. In Proceedings of the Paris International Scientific Meeting, Paris, France, 5 April 2019.
4. Jordan, T.F. Simple Proof of No Position Operator for Quanta with Zero Mass and Nonzero Helicity. *J. Math. Phys.* **1978**, *19*, 1382–1385. [CrossRef]
5. Fleming, G.N. Covariant Position Operators, Spin, and Locality. *Phys. Rev.* **1965**, *137*, B188–B197. [CrossRef]
6. Gordon, N.F. Nonlocal Properties of Stable Particles. *Phys. Rev.* **1965**, *139*, B963–B968. [CrossRef]
7. Hawton, M.; Baylis, W.E. Photon Position Operators and Localized Bases. *Phys. Rev. A* **2001**, *64*, 012101. [CrossRef]
8. Hawton, M. Photon Position Measure. *Phys. Rev. A* **2010**, *82*, 012117. [CrossRef]
9. Hawton, M.; Debierre, V. Photon position eigenvectors, Wigner's little group, and Berry's phase. *J. Math. Phys.* **2019**, *60*, 052104. [CrossRef]
10. Newton, T.D.; Wigner, E.P. Localized States for Elementary Systems. *Rev. Mod. Phys.* **1949**, *21*, 400–406. [CrossRef]
11. Chan, K.W.; Law, C.K.; Eberly, J.H. Localized Single-Photon Wave Functions in Free Space. *Phys. Rev. Lett.* **2002**, *88*, 100402. [CrossRef]
12. Creatore, C.; Andreani, L.C. Quantum Theory of Spontaneous Emission in Multilayer Dielectric Structures. *Phys. Rev. A* **2008**, *78*, 063825. [CrossRef]
13. Hoffmann, S.E. An Observable for the Separation Distance of Two Photons. *arXiv* **2021**, arXiv:2108.06671. [CrossRef]
14. Hawton, M. Photon Wave Mechanics and Position Eigenvectors. *Phys. Rev. A* **2007**, *75*, 062107. [CrossRef]
15. Bialynicki-Birula, I. On the Wave Function of the Photon. *Acta Phys. Pol. A* **1994**, *86*, 97–116. [CrossRef]
16. Maybee, B.; Hodgson, D.; Beige, A.; Purdy, R. A Physically-Motivated Quantisation of the Electromagnetic Field on Curved Spacetimes. *Entropy* **2019**, *21*, 844. [CrossRef]
17. Rueckner, W.; Peidle, J. Young's Double-Slit Experiment with Single Photons and Quantum Eraser. *Am. J. Phys.* **2013**, *81*, 951–958. [CrossRef]
18. Aspden, R.S.; Padgett, M.J.; Spalding, G.C. Video Recording True Single-Photon Double-Slit Interference. *Am. J. Phys.* **2016**, *84*, 671–677. [CrossRef]
19. Landry, A.; Hammad, F. Landau Levels in a Gravitational Field: The Schwarzschild Spacetime Case. *Universe* **2021**, *7*, 144. [CrossRef]
20. Park, S.; Park, B.; Nam, S.; Yun, S.; Park, S.K.; Mun, S.; Lim, J.M.; Ryu, Y.; Song, S.H.; Kyung, K.U. Electrically Tunable Binary Phase Fresnel Lens Based on a Dielectric Elastomer Actuator. *Opt. Express* **2017**, *25*, 23801. [CrossRef]
21. Shaltout, A.M.; Shalaev, V.M.; Brongersma, M.L. Spatiotemporal Light Control with Active Metasurfaces. *Science* **2019**, *364*, eaat3100. [CrossRef]

22. Lee, C.W.; Choi, H.J.; Jeong, H. Tunable Metasurfaces for Visible and SWIR Applications. *Nano Converg.* **2020**, *7*, 3. [CrossRef] [PubMed]
23. Damgaard-Carstensen, C.; Thomaschewski, M.; Ding, F.; Bozhevolnyi, S.I. Electrical Tuning of Fresnel Lens in Reflection. *ACS Photonics* **2021**, *8*, 1576–1581. [CrossRef]
24. Kim, J.; Seong, J.; Yang, Y.; Moon, S.W.; Badloe, T.; Rho, J. Tunable Metasurfaces Towards Versatile Metalenses and Metaholograms: A Review. *Adv. Photonics* **2022**, *4*, 024001. [CrossRef]
25. Novoselov, K.S.; Geim, A.K.; Morozov, S.V.; Jiang, D.; Zhang, Y.; Dubonos, S.V.; Grigorieva, I.V.; Firsov, A.A. Electric Field Effect in Atomically Thin Carbon Films. *Science* **2004**, *306*, 666–669. [CrossRef] [PubMed]
26. Bonaccorso, F.; Sun, Z.; Hasan, T.; Ferrari, A.C. Graphene Photonics and Optoelectronics. *Nat. Photonics* **2010**, *4*, 611–622. [CrossRef]
27. Polat, E.O.; Uzlu, H.B.; Balci, O.; Kakenov, N.; Kovalska, E.; Kocabas, C. Graphene-Enabled Optoelectronics on Paper. *ACS Photonics* **2016**, *3*, 964–971. [CrossRef]
28. Zhang, Z.; Qi, X.; Zhang, J.; Guo, C.; Zhu, Z. Graphene-Enabled Electrically Tunability of Metalens in the Terahertz Range. *Optics Express* **2020**, *28*, 28101. [CrossRef]
29. Chen, D.; Yang, J.; Huang, J.; Zhang, Z.; Xie, W.; Jiang, X.; He, X.; Han, Y.; Zhang, Z.; Yu, Y. Continuously Tunable Metasurfaces Controlled by Single Electrode Uniform Bias-Voltage Based on Nonuniform Periodic Rectangular Graphene Arrays. *Opt. Express* **2020**, *28*, 29306. [CrossRef]
30. Park, S.; Lee, G.; Park, B.; Seo, Y.; Bin Park, C.; Chun, Y.T.; Joo, C.; Rho, J.; Kim, J.M.; Hone, J.; et al. Electrically Focus-Tuneable Ultrathin Lens for High-Resolution Square Subpixels. *Light Sci. Appl.* **2020**, *9*, 98. [CrossRef]
31. Park, S.; Park, C.; Hwang, Y.J.; Kang, J.; Lee, G.; Seo, Y.; Chun, Y.T.; Rho, J.; Kim, J.M.; Hone, J.; et al. Focus-Tunable Planar Lenses by Controlled Carriers Over Exciton. *Adv. Opt. Mater.* **2021**, *9*, 2001526. [CrossRef]
32. Stauber, T.; Peres NM, R.; Geim, A.K. Optical Conductivity of Graphene in the Visible Region of the Spectrum. *Phys. Rev. B* **2008**, *78*, 085432. [CrossRef]
33. Geim, A.K. Nobel Lecture: Random Walk to Graphene. *Rev. Mod. Phys.* **2011**, *83*, 851–862. [CrossRef]

Disclaimer/Publisher's Note: The statements, opinions and data contained in all publications are solely those of the individual author(s) and contributor(s) and not of MDPI and/or the editor(s). MDPI and/or the editor(s) disclaim responsibility for any injury to people or property resulting from any ideas, methods, instructions or products referred to in the content.